Functional equation with twice iterated function.
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Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that satisfies to equation $f(f(x))+af(x)=bx$, where $a,b>0$.

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I. Lets try to find linear function that satisfies to the functional equation, namely let $f(x)=kx$, where $k>0$ because $f:\mathbb{R}^+\to\mathbb{R}^+$. Then $f(f(x))=k^2x$ and $f(f(x))+af(x)=bx$ becomes $k^2x+akx=bx,\forall x>0 \iff k^2+ak-b=0$.
Let $k_1=\dfrac{\sqrt{a^2+4b}-a}{2},k_2=\dfrac{\sqrt{a^2+4b}+a}{2}>k_1$, that is $k_1,-k_2$ be roots of the latter equation.

II. For any $n\in\mathbb{N}\cup\{0\}$ we define recursively function $f_n(x)$ by

$$\begin{cases} f_0(x):=x \\ f_n(x)=f(f_{n-1}(x)),n\in\mathbb{N} \end{cases}.$$

Then by replacing x in $f(f(x))+af(x)=bx$ with $f_{n-1}(x),n\in\mathbb{N}$ we obtain for $(f_n(x))$ the following recurrence

(**1**) $\quad f_{n+1}(x)+af_n(x)-bf_{n-1}(x)=0,n\in\mathbb{N}$.

We have $f_1(x)=f(x)=0\cdot x+1\cdot f(x), f_2(x)+af_1(x)=bx\iff f_2(x)=bx-af(x)$, $f_3(x)+af_2(x)=bf_1(x)\iff f_3(x)=bf(x)-a(bx-af(x))=-abx+(a^2+b)f(x),\ldots$
That is $f_n(x)$ can be represented in form of linear combination of x and $f(x)$, namely $f_n(x)=b_nx+a_nf(x),n\in\mathbb{N}\cup\{0\}$. Then by substitution $f_n(x)$ in such form in (**1**) we obtain $b_{n+1}x+a_{n+1}f(x)+a(b_nx+a_nf(x))-b(b_{n-1}x+a_{n-1}f(x))=0\Rightarrow$

(**2**) $$\begin{cases} b_{n+1}+a\cdot b_n-b\cdot b_{n-1}=0 \\ a_{n+1}+a\cdot a_n-b\cdot a_{n-1}=0 \end{cases}\iff\begin{cases} b_{n+1}=-a\cdot b_n+b\cdot b_{n-1} \\ a_{n+1}=-a\cdot a_n+b\cdot a_{n-1} \end{cases},\ n\in\mathbb{N}$$

with initial conditions $b_0=1,b_1=0,a_0=0,a_1=1$.
We have $b_2=b>0,b_3=-ab<0,a_2=-a<0,a_3=a^2+b>0$
and for any $n\in\mathbb{N}$ assuming $b_{2n}>0,b_{2n+1}<0,a_{2n}<0,a_{2n+1}>0$ we obtain using (**2**) $b_{2n+2}=-a\cdot b_{2n+1}+b\cdot b_{2n}>0,b_{2n+3}=-a\cdot b_{2n+2}+b\cdot b_{2n+1}<0$, $a_{2n+2}=-a\cdot a_{2n+1}+b\cdot a_{2n}<0,a_{2n+3}=-a\cdot a_{2n+2}+b\cdot a_{2n+1}>0$.
Thus, by MI for any $n\in\mathbb{N}$ holds $b_{2n}>0,b_{2n+1}<0,a_{2n}<0,a_{2n+1}>0$.

III. Since $f_n(x)>0$ for any $x\in\mathbb{R}^+$ then $b_{2n}x+a_{2n}f(x)>0\iff b_{2n}x>-a_{2n}f(x)\iff$ $f(x)<-\dfrac{b_{2n}x}{a_{2n}}$ and $b_{2n+1}x+a_{2n+1}f(x)>0\iff f(x)>-\dfrac{b_{2n+1}x}{a_{2n+1}},n\in\mathbb{N}$.
Thus, $-\dfrac{b_{2n+1}x}{a_{2n+1}}<f(x)<-\dfrac{b_{2n}x}{a_{2n}},n\in\mathbb{N}$.
Noting that $b_n=c_1k_1^n+c_2(-k_2)^n,a_n=d_1k_1^n+d_2(-k_2)^n$ and taking in account that $0<\dfrac{k_1}{k_2}<1$ we obtain $\lim\limits_{n\to\infty}\dfrac{b_n}{a_n}=\lim\limits_{n\to\infty}\dfrac{c_1k_1^n+c_2(-k_2)^n}{d_1k_1^n+d_2(-k_2)^n}=\lim\limits_{n\to\infty}\dfrac{c_1\left(-\frac{k_1}{k_2}\right)^n+c_2}{d_1\left(-\frac{k_1}{k_2}\right)^n+d_2}=\dfrac{c_2}{d_2}$.
Since $\lim\limits_{n\to\infty}\dfrac{b_{2n}}{a_{2n}}=\lim\limits_{n\to\infty}\dfrac{b_{2n+1}}{a_{2n+1}}=\dfrac{c_2}{d_2}$ then by Squeese Principle $f(x)=\dfrac{c_2}{d_2}x$.

From initial conditions $\begin{cases} c_1 + c_2 = b_0 = 1 \\ c_1 k_1 - c_2 k_2 = b_1 = 0 \end{cases}$ and $\begin{cases} d_1 + d_2 = a_0 = 0 \\ d_1 k_1 - d_2 k_2 = a_1 = 1 \end{cases}$

we obtain $\frac{c_2}{d_2} = k_1$ and, therefore, $f(x) = k_1 x = \frac{\sqrt{a^2 + 4b} - a}{2} x$.

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